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Review

Research, Application and Future Prospect of Mode Decomposition in Fluid Mechanics

Yun Long [1,*], Xi'an Guo [1] and Tianbai Xiao [2]

[1] National Research Center of Pumps, Jiangsu University, Zhenjiang 212013, China; gxa1998@163.com
[2] State Key Laboratory of High Temperature Gas, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, China; txiao@imech.ac.cn
* Correspondence: longyun@ujs.edu.cn

Abstract: In fluid mechanics, modal decomposition, deeply intertwined with the concept of symmetry, is an essential data analysis method. It facilitates the segmentation of parameters such as flow, velocity, and pressure fields into distinct modes, each exhibiting symmetrical or asymmetrical characteristics in terms of amplitudes, frequencies, and phases. This technique, emphasizing the role of symmetry, is pivotal in both theoretical research and practical engineering applications. This paper delves into two dominant modal decomposition methods, infused with symmetry considerations: Proper Orthogonal Decomposition (POD) and Dynamic Mode Decomposition (DMD). POD excels in dissecting flow fields with clear periodic structures, often showcasing symmetrical patterns. It utilizes basis functions and time coefficients to delineate spatial modes and their evolution, highlighting symmetrical or asymmetrical transitions. In contrast, DMD effectively analyzes more complex, often asymmetrical structures like turbulent flows. By performing iterative analyses on the flow field, DMD discerns symmetrical or asymmetrical statistical structures, assembling modal functions and coefficients for decomposition. This method is adapted to extracting symmetrical patterns in vibration frequencies, growth rates, and intermodal coupling. The integration of modal decomposition with symmetry concepts in fluid mechanics enables the effective extraction of fluid flow features, such as symmetrically or asymmetrically arranged vortex configurations and trace evolutions. It enhances the post-processing analysis of numerical simulations and machine learning approaches in flow field simulations. In engineering, understanding the symmetrical aspects of complex flow dynamics is crucial. The dynamics assist in flow control, noise suppression, and optimization measures, thus improving the symmetry in system efficiency and energy consumption. Overall, modal decomposition methods, especially POD and DMD, provide significant insights into the symmetrical and asymmetrical analysis of fluid flow. These techniques underpin the study of fluid mechanics, offering crucial tools for fluid flow control, optimization, and the investigation of nonlinear phenomena and propagation modes in fluid dynamics, all through the lens of symmetry.

Keywords: fluid mechanics; modal decomposition technology; proper orthogonal decomposition

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1. Introduction

In recent years, with the rapid development of computer technology, researchers can obtain more abundant flow field information through various technologies (such as Particle Image Velocimetry (PIV) technology, computational fluid dynamics (CFD), numerical simulation, etc.). However, it is very difficult to qualitatively analyze the complex and high-dimensional system that is obtained, and the computational cost is very high. Therefore, it is very important to reduce the dimensionality and the degrees of freedom in the original system, and it is necessary to replace the original high-dimensional complex model with a simplified model. At present, researchers have obtained some mature dimension reduction methods, such as the Proper Orthogonal Decomposition (POD) method, the Dynamic Mode

Decomposition (DMD) method, and the central manifold method. Among them, the POD method is more popular because of its wide range of applications and its strong accuracy.

Since the 1900s, modal decomposition techniques have achieved many representative results in fluid mechanics: as early as the 1930s, the Karman–Trefftz modal decomposition method decomposed the flow field into different modes. This method was widely used in the fields of aerodynamics and hydrodynamics at that time. Then in the 1970s, the Morf mode decomposition was obtained by combining the time-averaging technique with the Karman–Trefftz mode decomposition method. This method has replaced the Karman–Trefftz modal decomposition method in the analysis of turbulent flow fields and promotes the development of aircraft dynamics and environmental fluid mechanics. With the vigorous development of computer technology in the 1990s, POD can decompose the data set into specific modes. In fluid mechanics, the POD method can perform low-dimensional approximate analysis on the turbulent flow field, which greatly speeds up the calculation speed. It is used in combustion fluid mechanics and multiphase flow dynamics. It has also been widely used in other fields. Entering the 21st century, DMD uses the evolution characteristics of the flow field in time to decompose. This method enables a more accurate study of the characteristics of unstable flow and analysis of the noise phenomenon caused by the flow. Figure 1 [1,2] shows two application of POD and DMD in fluid mechanics

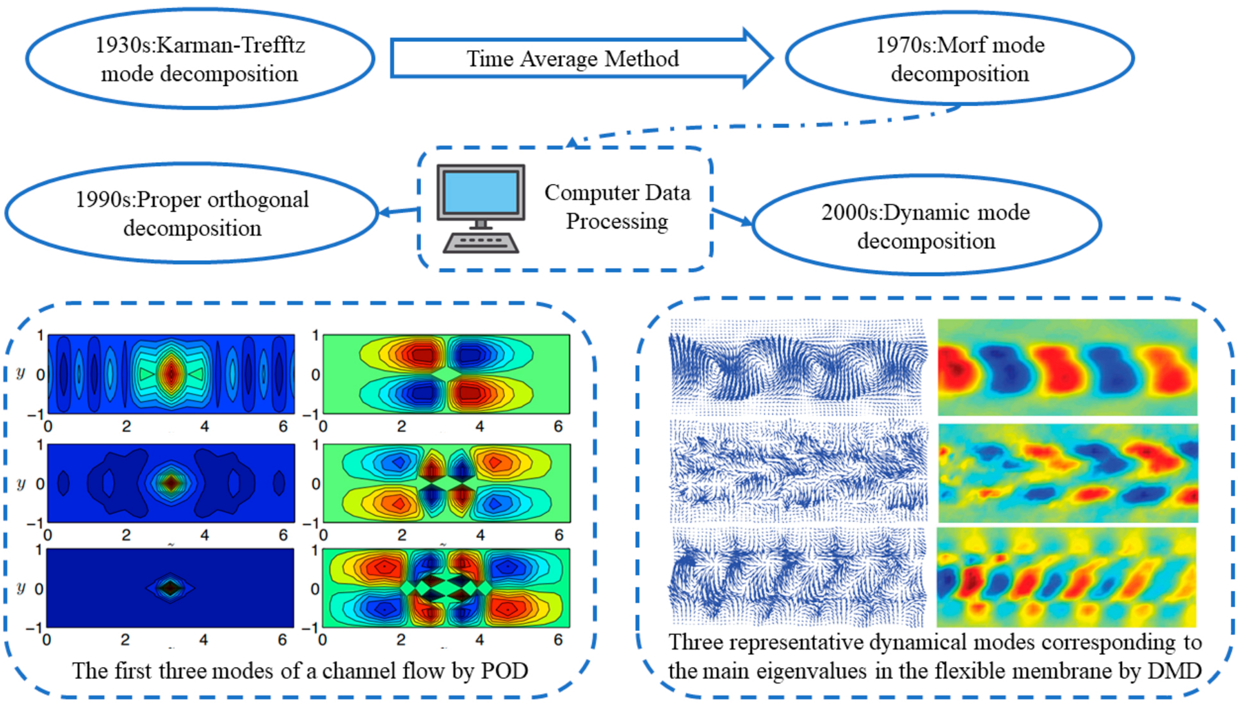

Figure 1. Development of modal decomposition methods in fluid mechanics [1,2].

The POD method is a mathematical tool for analyzing multi-dimensional data. The functionality of it is to describe the high-dimensional complex system in a low-dimensional approximate manner, to express the main features of the research target with fewer degrees of freedom, and then to achieve the purpose of simplifying the physical model, thereby saving computing time and computing load. The POD dimensionality reduction technology can perform the optimal low-dimensional approximation to the given data in the sense of fewest squares, so the POD method can efficiently solve the dimensionality reduction problem in the numerical simulation process of practical problems. POD is a method for processing large amounts of data, and it has good applications in flow field modal decomposition and model reduction. This method decomposes the flow field into different modes according to the difference in the energy level. Each mode is the product of an orthogonal basis function and a time function, which can be regarded as a time-varying flow structure with a specific energy amplitude. Its essence is that it projects the time-varying raw signal onto a set of time-varying superpositions of mutually orthogonal spatial signals. POD is widely used in engineering. The POD method has many aliases in different research fields, including KLD (Karhunen–Loève decomposition), KLE (Karhunen–Loève expansion), and PCA (principal component analysis). The POD method was first proposed by Pearson [3] in 1901 and then proposed again by Hotelling [4] in 1933. In addition, scholars in different fields, such as Kosambi, Karhunen, Pougachev, and Loeve, also proposed this method independently [5]. Subsequently, the POD method has been widely used in the field of fluid mechanics. In the study of fluid mechanics, the POD method is considered to be a powerful technical means to study turbulent flow and to understand the dynamic of complex flow mechanisms. In 1967, Lumley [6] applied the POD method to the field of turbulent flow research for the first time and identified identifiable flow structures with clear statistical periods and shapes in the flow field through orthogonally decomposing the space and velocity correlation functions. Most of the energy of the entire flow field is contained in the coherent structure, and this method is called the direct POD method. Lumley's distinctive POD turbulence research has aroused widespread attention and influence. However, in dealing with practical problems, the dimension of the spatial correlation matrix is very large, which seriously restricts the application of the POD method. In order to avoid the above problems, in 1987 Sirovich [7] improved the POD method and proposed the Snapshot POD method, which replaced the spatial correlation matrix with the temporal correlation matrix. This solved the problem of the huge spatial matrix caused by too many spatial points and enabled the calculation. The amount was greatly reduced, and the problem that could not be solved by the direct POD method before was solved. The improved Snapshot POD method is a landmark work product found in the Spectral Orthogonal POD correlation. Since then, the POD method has been more widely used and has been gradually applied to other fields besides fluid mechanics, such as modal analysis [8], stochastic structural dynamics [9], etc.

Since the POD mode may contain various flow structures with different frequencies, when flow field research needs to start from the mode with a lower frequency or energy, the mode obtained by the POD method alone is not enough to support the research [10]. Schmid proposed a DMD method in their study. Dynamic Mode Decomposition (DMD) is a highly general matrix decomposition technique based on singular value decomposition, and the low-rank structure and temporal features extracted from the DMD are associated with their temporal–spatial evolution. The importance of matrix decomposition methods and time series representations for data analysis can be understood if the influence of Principal Component Analysis (PCA) and the Fourier transform is considered. From a conceptual point of view, the DMD method has a rich history, stemming from Bernard Koopman's seminal work on nonlinear dynamical systems in 1931 [11]. However, due to the lack of computing resources in his day, theoretical development was largely limited. Interest in Koopman's theory was established by Mezić et al in 2004–2005 [12]. Schmid and Sesterhenn [2] and Seena [13] first defined DMD as an algorithm in 2008 and 2011, respectively. Rowley [14] quickly realized that the DMD algorithm is directly connected to the underlying nonlinear dynamics through the Koopman operator, opening the theoretical foundation for DMD theory. Much credit for the success of DMD can therefore be directly attributed to the pioneering contributions of Igor Mezic (UC Santa Barbara), Peter Schmid (Imperial College), and Clancy Rowley (Princeton University). The DMD method is not limited by the type of flow and can adopt experimental or numerical simulation results to analyze the dynamic characteristics of the flow field through directly extracting dynamic information from the data [15]. This method regards the flow field as a superposition of flow structures with different frequencies and calculates the modes and the eigenvalues of each order. Each DMD mode is independent of each other in time. It is precisely because of this that each mode calculated by the DMD method represents a unique frequency, so it performs better in dynamic linear analysis and periodic flow analysis [16]. Compared with the POD method, the DMD method can analyze the contribution of the flow structure to the flow field from the stability of the mode [17]. The widely used DMD method is developed based on the Koopman operator theory. The core of the Koopman theory is to transform the nonlinear system into an infinite-dimensional linear operator. Based on this, the DMD is extended to nonlinear systems and applied to the study of nonlinear flow [18] to express the change of certain data in the research object with time. Rowley et al. [2] identified that the DMD mode is a part of the Koopman mode and wrote the DMD calculation method through the friendship matrix.

2. Introduction to the POD Method

2.1. Classic POD Method

Intrinsic Orthogonal Decomposition (hereinafter referred to as POD) is a method derived from the statistical analysis of vector data, which is widely used in data dimensionality reduction, flow field analysis, etc. It involves a consideration of the m times when measuring the same phenomenon, and each measurement value is a vector containing a large number of n real number items $x_k, k = 1, 2, \ldots, m$ (where x_k is a digital image obtained in the kth trial). An important goal of the statistical analysis of the data is to discover interdependencies in the data and to reduce the data set to a $r \ll n$ with a smaller number of parameters. Mathematically, this situation can be described as an optimization problem [19]:

$$E|x - Px|^2 \to \min \tag{1}$$

S is a random real vector in the real number space R^n and E is the expected value. The essence of the above formula is to find the minimum mean square error. P is the projection operator of rank r, namely:

$$P = VU, UV = I_r \tag{2}$$

Assuming the matrix W, the above problem has a solution for the r main eigenvectors of the covariance array:

$$W = W_x = Exx' \tag{3}$$

Then, $\alpha_1^2 \geq \alpha_2^2 \geq \ldots$ are set to the ordered eigenvalues of W, and i_k is the corresponding orthonormalized eigenvectors, namely:

$$Wi_k = \alpha i_k, i_k' i_k = \beta \tag{4}$$

Then, the optimal orthogonal projection P is:

$$P = VV', V = [v_1\ v_2 \ldots\ v_r] \tag{5}$$

If, and only then, $\alpha_r > \alpha_{r+1}$.

With the deepening of the research in different fields and into actual needs, the POD method has also been improved in different directions. In addition to the traditional Classic POD, Sirovich proposed the Snapshot POD in 1987. This POD method improves the efficiency of solving matrix eigenvalues by reducing the order of the autocorrelation matrix and making it equal to the number of snapshots. The result is improved speed, and computation and simulation stability [20]. Both decomposition methods are used in engineering. When faced with the problem of data loss due to some reason, the Gappy POD method can be used to repair the incomplete samples. This method is improved based on the Snapshot POD, and the data can be completed by the least square method or iterative prediction.

2.2. Snapshot POD

The Snapshot POD method uses the linear combination of the original space function elements to represent the eigenmodes, and the order of the autocorrelation matrix is equal to the number of snapshots, which improves the solution efficiency. This method is also suitable for solving the problem that the matrix is difficult or impossible to solve, because the number of spatial points is more than the number of samplings points [21].

Construct the matrix U from the acquired target field data of n time nodes, then:

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \ldots & \boldsymbol{u}_n \end{bmatrix} \tag{6}$$

where $\boldsymbol{u}$ is the target field changing with time $\boldsymbol{u}(x,t)$:

$$\boldsymbol{u}(x,t) = \sum_{i=1}^{N} a_i(t)\boldsymbol{\Phi}_i(x) \tag{7}$$

To compute the POD basis, first compute the autocovariance matrix of the target field:

$$\boldsymbol{C} = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} \tag{8}$$

Then, solve for the eigenvalues:

$$\boldsymbol{C}A_i = \lambda_i A_i \tag{9}$$

Then, arrange the modes according to the eigenvalues λ from large to small to obtain the main modes:

$$\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_N = 0 \tag{10}$$

The POD mode can be constructed to obtain the time coefficient. If the flow field is regarded as the superposition of the basic flow and the flow field pulsation, it is necessary to remove the matrix formed by the time average value from the obtained sample matrix and to calculate the pulsation matrix and the correlation matrix with its eigenvalues and eigenvectors.

Due to the limitations of the principle of the Snapshot POD, the accuracy of the data acquisition and flow field reconstruction from this method depends on the collected snapshot samples, so the samples are required to cover a wide range of research objects [22]. In terms of usage, the Classic POD method is mainly used to analyze the correlation between different research points in the instantaneous state, and the Snapshot POD method often focuses on the correlation analysis of multiple images distributed over time at the same location [23].

2.3. Gappy POD

The Gappy POD method can recover partially lost original data in various ways. The following is a brief introduction to the Classic Gappy POD method [24].

For research with sample vector data loss, if the sample $\boldsymbol{U}$ number is n, then:

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{u}_2 \end{bmatrix}^T \tag{11}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 \end{bmatrix} \tag{12}$$

$$\boldsymbol{U}_1 = \sum_{j=1}^{n} a_j\boldsymbol{\Phi}^j \tag{13}$$

Only one of them, $\boldsymbol{u}_2$, has data loss, and $\boldsymbol{u}_2$ is:

$$\boldsymbol{u}_2 = \begin{bmatrix} \boldsymbol{u}_{21} & \boldsymbol{u}_{22} \end{bmatrix} \tag{14}$$

In the formula, $\boldsymbol{u}_{22}$ is the missing data, and $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ are the POD base coefficients of $\boldsymbol{u}_{21}$ and $\boldsymbol{u}_{22}$, respectively. Complementary data can be obtained by combining the singular value calculation of the matrix $\boldsymbol{U}$ with the least square method. However, the POD base coefficients obtained in this way have certain errors and are generally calculated in other improved ways, such as through iterative prediction and completion [25].

3. Research on Mode Decomposition Methods

3.1. Mode Decomposition in Aerodynamics

Mode decomposition in the study of the airfoil flow field is an important analytical tool, which helps us to understand the complexity of aerodynamic problems and the characteristics of airfoil flow. The role of modal decomposition is reflected in the following four aspects:

Identifying and Classifying Modes: Mode decomposition can decompose complex airfoil flow problems into simple fundamental modes, each corresponding to a unique flow structure. By identifying and classifying these modes, we can more precisely describe different flow phenomena such as vortex structures, wakes, boundary layers, and pressure distributions.

Modal Feature Extraction: Modal decomposition enables us to extract information regarding key features from complex flow field data. Each mode represents an important feature in the flow: for example, the cutoff mode can reflect in which frequency range the main energy of the flow is concentrated. By analyzing these modes and the relative contributions between them, we can better understand the dynamic properties of airfoil flow.

Flow Field Reconstruction and Prediction: Through modal decomposition, we can represent flow field variables as a linear combination of individual modes. This representation can be used to reconstruct the entire flow field, allowing us to understand the details of the flow in greater detail. In addition, the model based on modal decomposition can also be used for the prediction and simulation of the flow field. Through controlling the evolution of the modal, we can predict the flow response under different operating conditions, which is of great significance for optimizing the design and performance of the airfoil.

Physical Mechanism Explanation: Modal decomposition provides us with a physical mechanism explanation of airfoil flow. By analyzing the amplitude and phase of each mode, as well as the interaction between modes, we can reveal various mechanisms in the flow, such as vortex–vortex interactions, the coupling of pressure distributions, boundary layer development, etc. The explanation of this physical mechanism contributes to a deeper understanding of the airfoil flow and provides guidance for improving airfoil design and optimizing flow control.

In summary, modal decomposition plays a vital role in the study of airfoil flow fields. It can not only help us to identify and classify flow structures, extract feature information, and reconstruct flow fields and predict flow responses but it can also reveal the physical mechanism of the flow and promote the further development of airfoil design and flow control.

3.1.1. Research on Modal Decomposition in Aerodynamics

At present, hypersonic vehicles are in a period of vigorous development at home and abroad, and the optimization and design of the aerodynamic characteristics of the aircraft airfoil cannot be separated from the CFD numerical calculation. Nie Chunsheng et al. [26] took the shape of the Hermes aircraft as the research object, obtained a three-dimensional thermal environment database using the CFD numerical method, and used the POD method to reduce the CFD database. Combining the POD method and RBF interpolation, a suitable complex thermal environment database was established. The surrogate model (Surrogate Model, SM) for predicting the heat flow on the shape surface can quickly predict the surface thermal environment parameters that meet the accuracy requirements in an unknown state. On the premise of not losing prediction accuracy, this method can greatly improve the calculation efficiency and effectively make up for the shortcomings of engineering algorithms. Sun Chong et al. [27] carried out a numerical simulation on the static stall and dynamic stall of the S809 airfoil at an angle of attack of 20°. They studied the unsteady flow field around the wing and extracted the static stall and dynamic stall unsteady flow field by using the POD method. The main flow modes of the pressure field were combined with the POD coefficients of the corresponding modes to analyze the unsteady flow, which is of great significance to the study of the stall problem of the wind turbine airfoil. Figure 2 [26,28] shows that research on POD in aerodynamics.

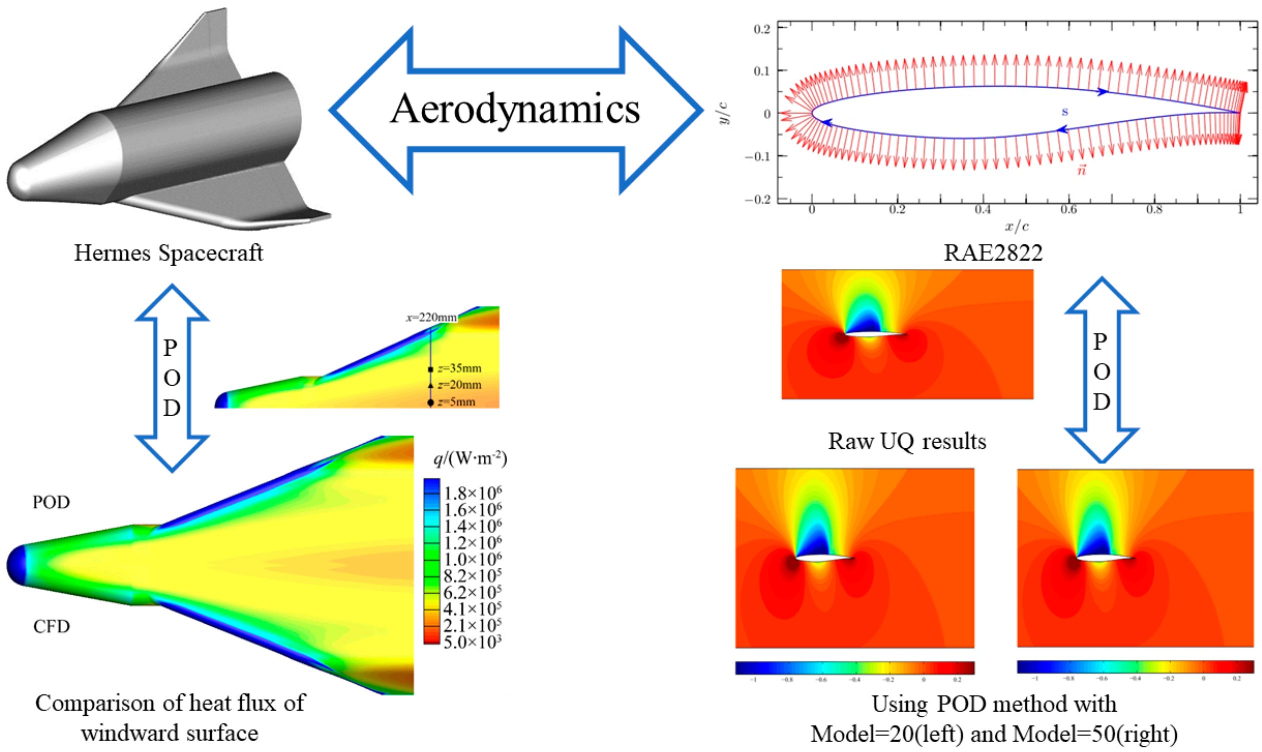

Figure 2. Application of POD in aerodynamics [26,28].

Yang [29] established a model based on the B737-200 aircraft cockpit and carried out research on the flow field characteristics in the aircraft cockpit under different air supply conditions. The influence of flow field characteristics on pollutant transmission was analyzed. Li Bo et al. [30] also proposed an efficient adaptive sequential optimization method based on the POD surrogate model and compared and analyzed the two-dimensional airfoil design optimization. It was found that this method can significantly improve the optimization efficiency and has a higher speed. Compared with the time-consuming calculation of the CFD, the calculation amount of the whole calculation process is also very small, which provides an effective solution for the fitting or optimization of the field volume data. Arash Mohammadi et al. [28] applied the POD method to uncertainty quantification (Uncertainty Quantification, UQ) analysis, combined the POD method with Compressed Sensing (Compressed Sensing, CS), and applied this method to the RAE2822 airfoil designed by NASA. In the UQ analysis of the CFD simulation of the transonic turbulent flow field around the rotor, the computational cost was significantly reduced.

3.1.2. Application of Mode Decomposition in Aerodynamics

Liu Zhao et al. [31] established the aerodynamic model of wake excitation under the conditions of periodic variation and sinusoidal variation pressure and proved that the error between the CFD calculation results and the reduced order model obtained by the POD method was very small. Liu Nan et al. [32] chose the POD–Kriging model, combined with the constraint processing method, to reconstruct the aerodynamic shape design space, which also improved the design efficiency while ensuring a high accuracy. The POD method was applied to the study of the temporal–spatial evolution of the supersonic mixing layer under different working conditions to obtain the energy modal distribution, the time evolution characteristics, and the frequency domain characteristics of the modal coefficients, as well as the modal space structure [33]. Carlos Quesada et al. [34], for the first time, used the POD dimensionality reduction technique to predict the deformation of microcapsules flowing through straight microfluidic channels in a steady state. Zhang Yang [35] used the POD and DMD methods to compare the unsteady characteristics in the study of the high angle of attack flow of an aircraft and found that the POD mode contains movements of multiple frequencies and that the more dominant the POD, the greater the energy contained in the mode. The flow field reconstruction efficiency was higher, and the main feature of the motion extracted by the DMD mode was a single-frequency mode. Each mode was more stable. Figure 3 [33,34] shows that application on POD in aerodynamics. Table 1 shows the study of mode decomposition in aerodynamics.

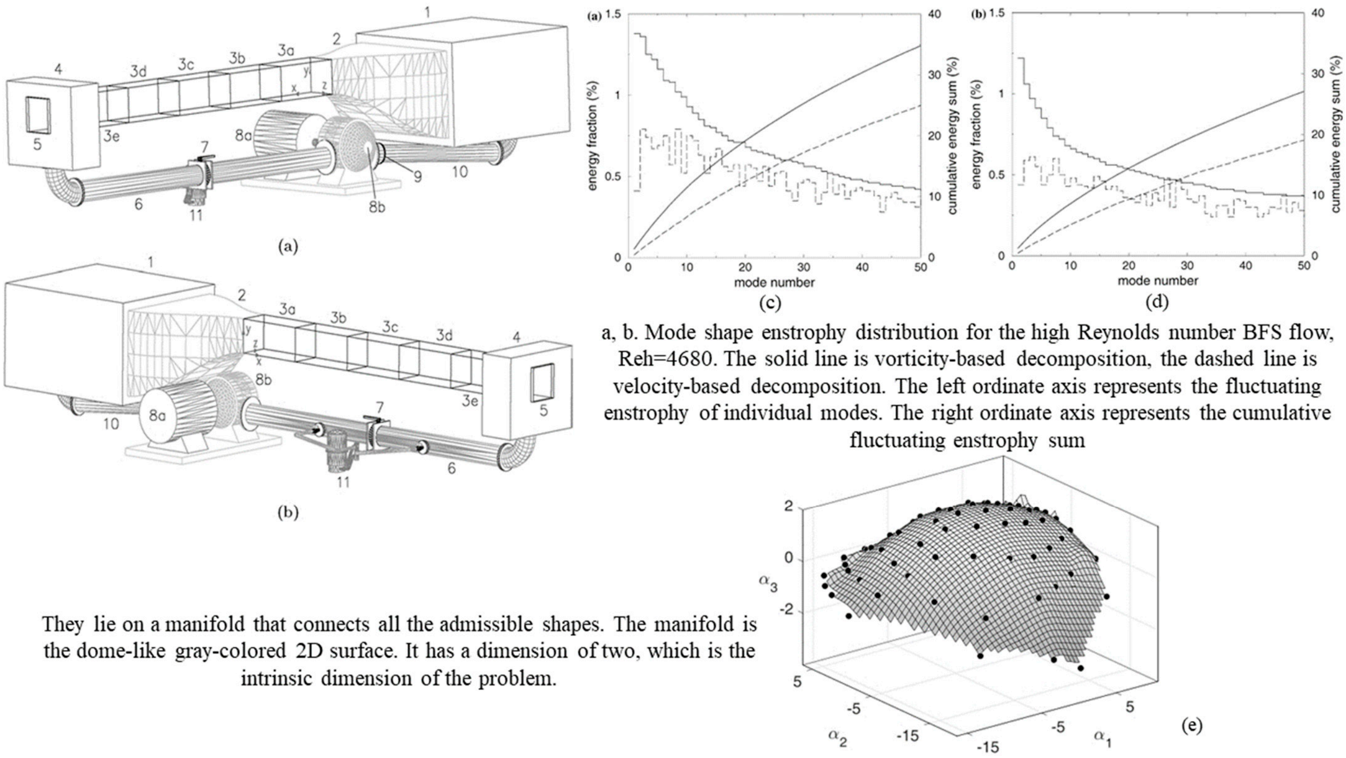

Figure 3. Energy modal distribution and modal space structure obtained by POD [33,34]. (**a**) Region I. (**b**) Region II. (**c**) Region I modes. (**d**) Region II modes. (**e**) Representation of the coefficients (α1, α2, α3) of all the capsule shapes contained in C (black dots).

Table 1. The study of mode decomposition in aerodynamics.

Authors	Research Contents and Applications	Methods
Nie Chunsheng [26]	Using the POD method to reduce the order of the CFD 3D thermal environment database	POD method and RBF proxy model
Sun Chong [27]	Extraction of main features of static stall and dynamic stall for unsteady flow field around the airfoil using the POD method	POD method
Arash Mohammadi [28]	UQ Analysis of transonic turbulent flow field	UQ analysis, compressed sensing, and the POD method
Pei Chunbo [29]	Analysis of the flow characteristics in an aircraft's interior cabin under different working conditions	POD method

Table 1. *Cont.*

Authors	Research Contents and Applications	Methods
Li Bo [30]	Optimized for 2D airfoil design	POD method and intelligent optimization algorithm
Liu Zhao [31]	Comparing CFD results with POD results	POD method
Liu Nan [32]	Aerodynamic shape design optimization	POD–Kriging method
Carlos Quesada [34]	Deformation during steady state flow through a straight microfluidic channel	POD method and small-scale numerical calculation method
Zhang Yang [35]	Comparison of POD and DMD at a high angle of attack of the aircraft	POD method and DMD method

3.2. Mode Decomposition in Hydrodynamics

CFD (computational fluid dynamics) are widely used in the calculation of various flow fields because of their high calculation accuracy and few restrictions [36], but they also have the disadvantage of a long calculation time. In the practical application of CFD, if the research or experiment requires multiple iterations, the choice of the CFD method can lead to a large amount of calculations, resulting in significant requirements for the performance of the computer [37]. Moreover, if the data acquisition is only based on CFD or basic experimental flow field analysis, the data information components can be very complicated, the physical characteristics of the flow field cannot be obtained, and it is even more difficult to determine the flow field characteristics that mainly affect the flow state change in the flow field. Modal decomposition can reduce the dimensions of the calculation, extract the main features, and reduce data redundancy. Through modal decomposition, the complex flow field can be decomposed into basic modes. The independent calculation of each mode is an effective method to improve the efficiency of the CFD analysis.

3.2.1. Research on Mode Decomposition in Hydrodynamics

Jia Xuyi et al. [38] identified a fast calculation method of the flow field based on intrinsic orthogonal decomposition. A backpropagation neural network was proposed. In the construction of the POD and BPNN models, partition and cluster sampling strategies were introduced to improve the modeling efficiency and to reduce the time-consuming model training, respectively. The results of the steady flow field case of the variable geometry airfoil show that, in the case of subsonic speed, the trained model can guarantee the prediction accuracy of the isobar, the airfoil pressure coefficient, and other information in the flow field. The average prediction error of the lift–drag coefficient was 0.4%. In the case of transonic velocity, the average prediction error of the lift–drag coefficient of the model obtained from the training was within 1.4%, and the shock wave position was also predicted more accurately, according to Li Dali et al. [39]. Optimized calculations with structural grids also simplify the mathematical model by constructing a reduced order model (ROM) of the unsteady flow field. The POD is used to calculate the orthogonal basis with the smallest error in the function space—the optimal orthogonal basis—and to project the original high-dimensional flow field data to the low-order vector space through the optimal orthogonal basis. The flow field can be transformed according to the difference in energy, which is decomposed into various modes, so that the original target flow field can be expressed in several dimensions in an approximate manner. Its eigenvalues can represent the energy of the corresponding POD mode while retaining its main features and flow structure [40]. When using the POD method to process the data, L. Lathauwer et al. [41] found that, when the singular value decomposition method is used to decompose the eigenvalues, its calculation efficiency is higher than that of the traditional calculation method, and more accurate high-order modes can be obtained. The intrinsic orthogonal decomposition method has high calculation efficiency and better accuracy.

3.2.2. Mode Decomposition and Reconstruction, and the PIV Experiment in Hydrodynamics

Wang Zangang [42] used the POD method to analyze the flow field around a two-dimensional square column under low Reynolds number conditions and found that the energy of the first four POD modes accounted for more than 99%. The main low-order modes in the POD contain higher energy, which facilitates a faster reconstruction of the flow field [43]. Shady E. Ahmed et al. [44] clearly showed the dominant structure of the original flow field after denoising through the flow field reconstructed by the POD. The purpose of the POD low-dimensional reconstruction of the flow field was to study the nature of the flow field without interference from small-scale eddies. Zhang Zhengchuan et al. [45] took the centrifugal pump as an example to explore the transient flow field inside the centrifugal pump and verified that the low-order POD mode represents the low-frequency, large-scale flow in the flow field, which contains high energy and can show the main flow characteristics. High-order modes correspond to high-frequency, small-scale flows with low energy, and, with the gradual increase of the POD mode order, the corresponding flow structures also show complex characteristics. Fu Jue et al. [46] explored the unsteady flow of the blade tip clearance flow field in the near-stall state, adopted the POD method to analyze the flow field flow and to reconstruct the flow field, and found that the first-order POD mode can restore the flow field. They discovered that the main features of the flow field were that, with the increase of the POD modal order, the reconstructed flow field was closer to the actual flow field, and the details in the flow field were expressed more clearly and accurately.

Qin Hao et al. [47] used the POD method to decompose the transient velocity field of the flow field around the column at a large Reynolds number obtained by the particle image velocimetry technique (PIV) and recalculated the flow field based on the first six modes of the POD mode. The experiment shows that the first six modes indicate the main characteristics of the flow field, and the flow field can be reconstructed more accurately. Bi et al. [48] used stereovision and PIV technology to identify the trajectory and flow field characteristics of a free-falling annular disk and extracted the coherent structure of the wake behind the disk with the POD method. Through decomposing the flow field into the different POD modes, they clarified the reason the HH (Hula-Hoop) motion and the HM (Helical Motion) motion [49] are different. S. Kumar et al. [50] used Laser Doppler Velocimetry (Laser Doppler Velocimetry, LDV) and PIV Technology. The flow field in the draft tube of the Francis turbine model was studied, and the POD method was used to analyze 250 particle image snapshots, including both axial and radial snapshots. Song Yuchen et al. [51] used a time-resolved tomo-PIV to study the non-uniform flow above the reactor coolant pump. The three-dimensional velocity and pressure of the non-uniform inflow were reconstructed with a time-resolved tomographic particle image velocimetry to evaluate its effect on the RCP. Figure 4 shows that there are two large-scale vortices in the non-uniform inflow below the SG, and the size of the small-scale turbulent vortices decreases with the flow rate. Large-scale vortices exist in the form of alternating counter-rotating pair vortices (M–M vortices). The pressure field reconstructed from the three-dimensional velocity of the POD was used to study the influence of the non-uniform inflow of the RCP. The first 412 modes occupy 90% of the energy region. Table 2 shows the study of mode decomposition in hydrodynamics.

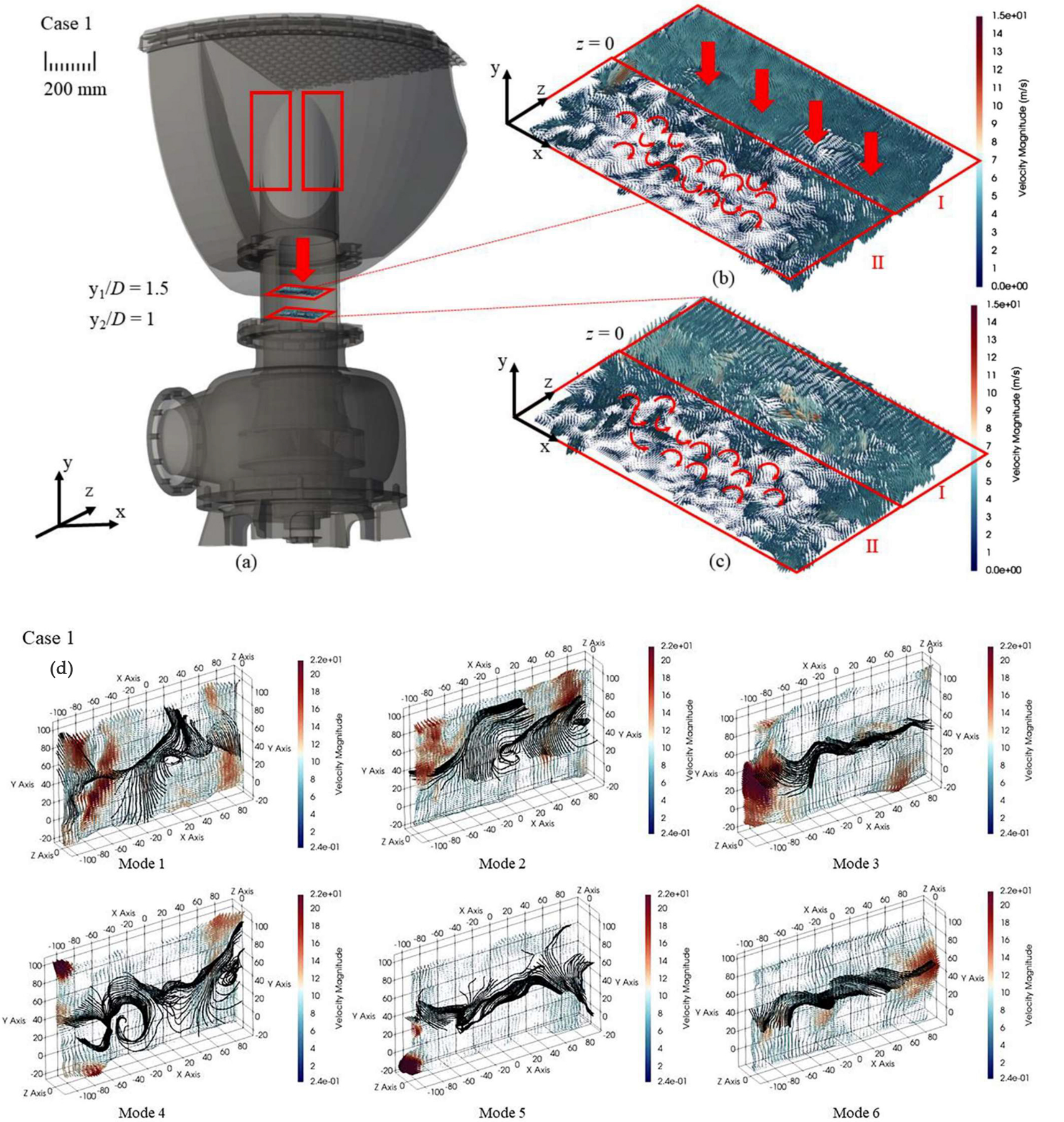

Figure 4. (**a**) Circumferential non-uniform velocity field of case 1 and 3d geometrical model; (**b**) the 3D vector at the plane of y1/D; (**c**) the 3D vector at the plane of y2/D; and (**d**) the spatial distribution of the POD modes in R1 for cases 1 [51].

Table 2. Study of mode decomposition in hydrodynamics.

Authors	Research Contents and Applications	Methods
Li Dali [39]	Structural grid optimization calculation	ROM reduction model and POD method
Wang Zanggang [42]	Analysis of flow around two-dimensional square column by POD method	POD method
Shady E. Ahmed [44]	POD reconstructs the original flow field after denoising	POD method
Qin Hao [47]	Transient velocity field of flow around a column under large Reynolds number obtained by POD decomposition of PIV	POD method and PIV technology
Song Yuchen [51]	Hydrodynamics characteristics of non-uniform inflow in RCP by POD and PIV	POD method and PIV technology

3.3. Mode Decomposition Methods in Other Fields

POD also has other types of deformation and development, such as SPOD, BPOD, MSM, etc. [52]. With the deepening of research and changes in actual needs, the pure POD method is not enough to achieve the purposes of research or application. Interdisciplinary and cross-field engineering and projects have put forward higher requirements for research methods. The POD agent model is in this context. It is one of the products produced in the environment of traditional classic proxy models, which include the radial basis function model (RBF), the polynomial response surface model (PRS), the artificial neural network (ANN), the Kriging model (KRG), etc. [53]. They can simplify the difficulty of solving in the face of numerical calculation and simulation with a huge amount of data. However, the traditional proxy model only considers the relationship between scalar data and ignores the data related to the characteristics of the flow field itself. Therefore, the traditional proxy model does not have the ability to describe the complete flow field.

3.3.1. Mode Decomposition in Nonlinear Vibration Systems

The POD surrogate model combines system identification and smooth feature extraction methods, can comprehensively consider the data of the flow field without modeling all data points, and further improves the calculation efficiency on the basis of the traditional surrogate model. Li Bo et al. [30] verified and analyzed the accuracy of the POD proxy model, calculated the POD proxy model under different basis function orders, and used the energy information capacity to represent the accuracy. They found that the accuracy of the POD proxy model was higher than that of the proxy model, and the calculation time was shorter than the traditional direct optimization method. Zhang Lingfeng et al. [54] combined the POD, CB, and ROM to obtain the reduced order model (PCB-ROM) of the overall structure, which can correctly reflect the change of the flow field caused by the change of some parameters and greatly improve the calculation efficiency. Mei Guanhua et al. [55] used the POD-ROM corresponding to chaos to analyze the flutter of the two-dimensional wall panels. The calculation accuracy of this method is very close to that of the traditional method, but it has a higher calculation efficiency. Li Kui et al. [56] modeled the stratospheric wind field in Changsha based on the POD and found that the energy of the first five modes accounted for 98.9% of the total energy. Multidisciplinary integration is the source of innovation. The POD method is also widely used in structural dynamics. In the early 21st century, the POD method was gradually applied to structural dynamic systems. Deng Zichen et al. [57] first established the dynamic equation of a flexible cantilever beam impact system based on the Euler–Bernoulli principle and then successfully applied the POD method to the order reduction process of the impact system. Numerical results show that the method is feasible and has a high efficiency, which lays a foundation for the study of system control. In 2020, Zhao Yang et al. [58] proposed a new model reduction method for the beam structure dynamic model. The purpose of greatly reducing the number of degrees of freedom is achieved by reducing the basis vector. Lu Kuan et al. [59] improved the traditional POD method based on the inertial manifold theory, perfected the transient POD method, and proposed the method of determining the optimal dimensionality reduction model of the original high-dimensional complex system based on the physical meaning of the transient POD method. The improved POD method is used to reduce the dimensions of the original rotor-bearing system. Through a comparison of the phase diagram, the axis trajectory, the amplitude–frequency curve, and bifurcation characteristics before and after the dimension reduction, it was found that the simplified model after dimension reduction retains the original model well, including the dynamic characteristics. Li Yuwei et al. [60] obtained the node displacement field of the grid-stiffened shell model through static analysis and assembled it into a snapshot matrix and then used the POD technology to extract the principal components of the snapshot matrix as the transformation matrix. In order to achieve a realized model reduction, the POD method has also been used to study the dynamic order reduction problem of the cantilever plate geometric nonlinear structure, which can improve the solution efficiency of structural nonlinear dynamic systems under the condition of large geometric deformation [61]. Li [62] proposed a hybrid reduced order model combining the POD method and discrete empirical interpolation method (Discrete Empirical Interpolation Method, DEIM), which is used to accelerate the simulation of a single-phase compressible gas flow in porous media in petroleum engineering. The reconstruction, prediction accuracy, and calculation speed of the POD-DEIM method have been verified via examples. Subsequently, Chutipong Dechanubeksa et al. [63] continued to optimize this method and introduced a DEIM correction method based on the concept of GPOD (Gappy Proper Orthogonal Decomposition) [64], called the POD-GPOD method. This method takes into account the accuracy of the POD method and the efficiency of the POD-DEIM method. Figure 5 shows that the POD and DMD methods in other fields [61,65,66].

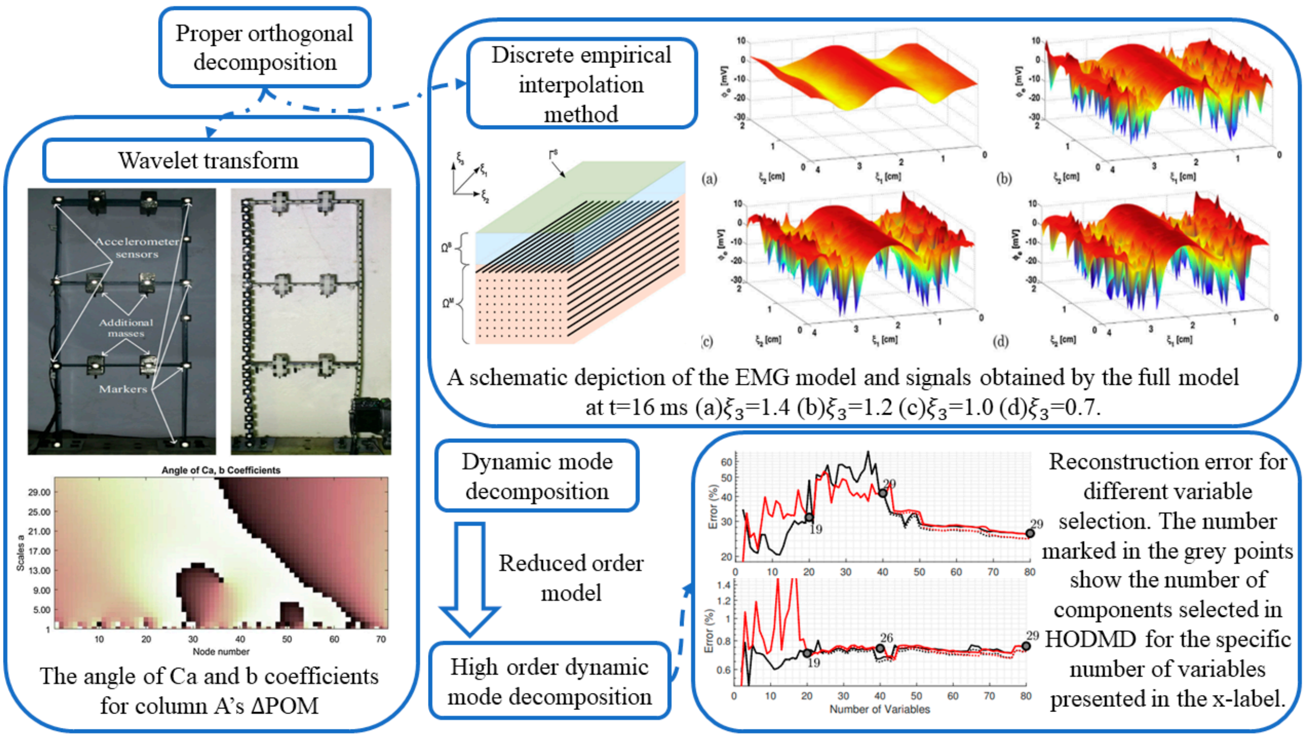

Figure 5. Application and promotion of POD and DMD methods in other fields [61,65,66].

The POD method combined with wavelet transforms can accurately locate the damaged part of the steel frame, and this method can detect the damage of the bending moment connection within an acceptable range of accuracy [66].

3.3.2. Mode Decomposition in Machine Learning

Wei Yun [67] proposed for the first time to embed genetic algorithms and the adjoint method in the POD method, completed the multi-method integrated design of the artificial environment, improved the calculation efficiency of the reverse design on the basis of ensuring calculation accuracy, and realized the artificial environment based on CFD. Efficient global optimization reverse design, based on the POD method, can establish a fast ice shape prediction reduction model [68]. This method can be applied to single- and multi-parameter ice shape predictions and has a reference value for the application of large parameter research. It provides an effective and feasible method for icing flight test certifications and ice-accommodating designs. The POD method is applied to the space–time evolution of supersonic mixed layers under different operating conditions to obtain the time evolution characteristics of the energy modal distribution, as well as modal coefficients, frequency domain characteristics, and the modal space structure [69]. Lv Xiaolong et al. [70] established a POD-RBF proxy model with a good inversion accuracy and generalization performance, and a fast iterative update inversion algorithm, using the POD method to extract the intrinsic vector and the RBF method to interpolate to obtain the surrogate model of the finite element model. At the same time, a new high-efficiency iterative inversion method can be established by combining the global optimization ability of the particle swarm algorithm and the fast local convergence advantages of the Gauss–Newton method. Ahmed Rageh et al. [71] further improved the POD-ANN method in order to study the fatigue and corrosion defects of rivet joints, combining the POD mode extracted from the measured structural response and the orthogonal mode calculated from the numerical model. As a result, the POD-ANN method is a robust fatigue damage identification tool for railway steel bridges. Figure 6 shows the POD-ANN field investigation flowchart.

Staf Roels et al. [72] used the POD method and the Proper Generalized Decomposition (PGD) method to study the heat transfer phenomenon of large-scale masonry walls, and the results showed that the POD method can provide more accurate results. Chinchun Ooi et al. [73] combined several machine learning models with the POD method to obtain the classic result of simulating the flow around a stationary cylinder. The method first calculates the POD or DMD mode and the time coefficient of the simulated data (with a snapshot), and then uses the long short-term memory network (Long Short-Term Memory, LSTM) model to predict the time coefficient of the mode [74]. Experiments show that the model has a high accuracy for predicting the time coefficients of snapshots of Kelvin–Helmholtz instabilities and mass diffusion problems. Steffen Kastian et al. [75] proposed the adaptive POD method (APOD), which introduced the principle of selecting snapshot subbases for constructing matching bases, which can give more accurate approximations.

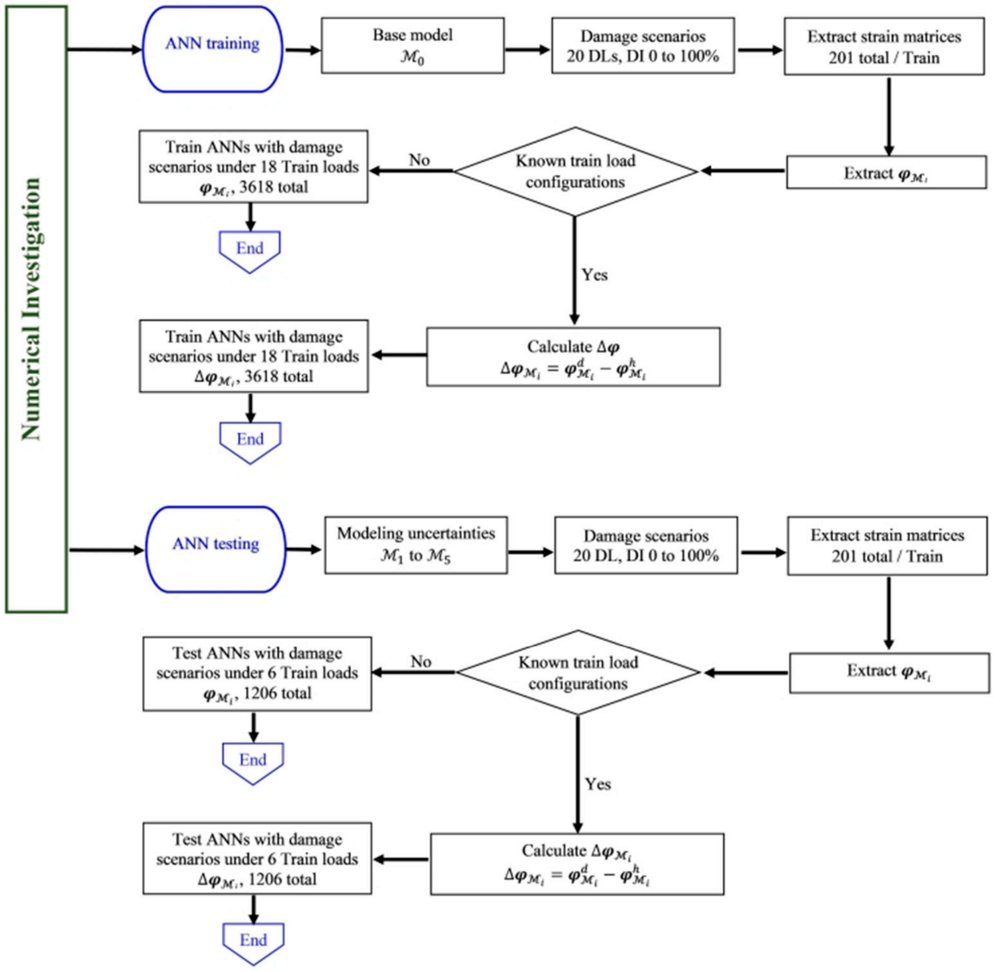

Figure 6. Field investigation flowchart [71].

3.4. Dynamic Mode Decomposition and the Order Reduction Model

The use of POD alone is not enough to support the construction of a flow field model to analyze the change of the flow field. It needs to be combined with other methods to establish a model based on the modal after POD processing, such as the POD-Galerkin method mentioned above. This is a dimensionality reduction model. The eigenvalues of each mode of the DMD can represent the change of the flow field flow, and the main characteristics of the flow field in time and space can be obtained at the same time. Modal stability analysis is also one of the unique features of DMD compared to POD [12]. In addition, the modes obtained after the POD calculation include flows of various frequencies, which are very complicated when analyzing physical phenomena. This is not conducive to the dynamic analysis of the flow field. For example, when studying transonic buffeting, the POD method is likely to ignore the flow field. Those components with a dominant frequency but low energy have a negative impact on the flow field analysis and reconstruction [76]. Different modes of DMD have different frequencies, and each mode has only a single frequency, which is convenient for analyzing the flow characteristics of the flow field at different frequencies [64].

3.4.1. DMD Method

The following is a brief introduction to DMD with the approximate matrix method as an example [77].

Assuming that there are n snapshots or data samples arranged in time through experiments or data simulations, the time interval between two adjacent snapshots is $\boldsymbol{\Delta t}$, and the data obtained by n snapshots are each a column vector $\boldsymbol{x_i}$, a data set can be obtained as follows:

$$\{x_1, x_2, x_3, \ldots, x_n\} \tag{15}$$

When the number of snapshots is sufficient, the rank of the above matrix will remain unchanged. At this time, there is the following:

$$x_{i+1} = Ax_i \tag{16}$$

where $\boldsymbol{A}$ is the system matrix of the high-dimensional flow field, also known as the mapping matrix. Then, the data in the data set is divided into two sets containing $n-1$ data. These are as follows:

$$X_1^{k-1} = [x_1, x_2, x_3, \cdots, x_{k-1}] \tag{17}$$

$$X_2^k = [x_2, x_3, x_4, \cdots, x_k] \tag{18}$$

$$X_2^k = AX_1^{k-1} \tag{19}$$

Since X_1^{k-1}, X_2^k is not a square matrix, and the dimension of $\boldsymbol{A}$ is high, $\boldsymbol{A}$ cannot be obtained directly. As a result, it is necessary to obtain an approximate matrix for $\widetilde{A}$ and, at the same time, use X_1^{k-1} to carry out a singular value decomposition, with the result that:

$$X_1^{k-1} = USV^{\mathrm{H}} \tag{20}$$

$$A = U\widetilde{A}U^{\mathrm{H}} = U^{\mathrm{H}}X_2^kVS^{-1} \tag{21}$$

Therefore, the DMD mode and its corresponding mode coefficient can be obtained as follows:

$$\Phi = X_2^kVS^{-1}W \tag{22}$$

$$b = \Phi^{-1} \cdot x_1 \tag{23}$$

The above formula shows that the mode coefficient can show the influence degree of the mode in the flow field.

DMD can perform modal stability analysis. The calculation method is to obtain the magnification and frequency of the mode by approximating the eigenvalues corresponding to each eigenvector of the matrix, where the real part represents the magnification and the imaginary part represents the frequency. If the magnification is negative, it means that the mode is stable; otherwise, it corresponds to an unstable mode. If the magnification is zero, it means that the mode is a periodic mode [78]. In addition, the modal stability can also be judged by the distribution of the eigenvalues on the unit circle [79]. Among them, the horizontal axis represents the real part of the eigenvalue, and the vertical axis represents the imaginary part of the eigenvalue. If the point of the eigenvalue is outside the unit circle, it means that the magnification is positive; otherwise, it is negative. If the point is on the unit circle, it means that the magnification is zero. DMD has been applied in many fields by virtue of its excellent characteristics, but it still has great mining potential. Furthermore, various deformation and improvement methods have appeared, such as sparsely enhanced DMD (SPDMD, sparsity-promoting DMD), adaptive non-uniform sampling DMD (NU-DMD, non-uniform DMD), optimal mode decomposition OMD (optimal mode decomposition), etc. [80].

3.4.2. Mode Decomposition and the Order Reduction Model

The ROM model provides a way of thinking. This model can express most of the flow field characteristics in the high-dimensional unsteady flow field while ensuring that the precision is reduced for modeling, reducing the time consumed by the calculation and the memory occupied. This reduced order model is mainly divided into two categories, including the ROM method for the system identification method development and the

ROM method for the flow field feature extraction. The former includes the ARX model, the Vikterra series method, and the neural network model [81]. Although the input and output relationship of the target system can be obtained, the essence of the system cannot be analyzed. In the latter, intrinsic orthogonal decomposition (proper orthogonal decomposition) and dynamic mode decomposition are two commonly used methods, namely POD and DMD [82]. Figure 7 shows mode decomposition and the order reduction model in hydrodynamics [76,82].

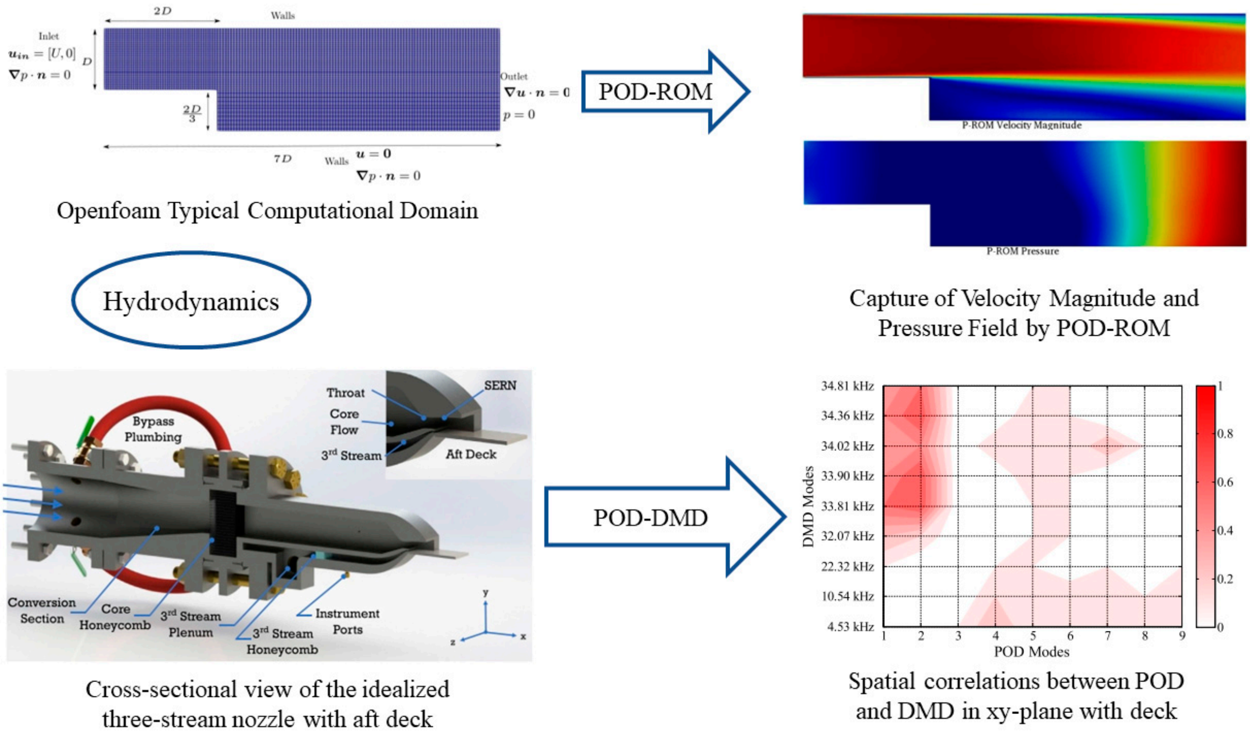

Figure 7. Modal decomposition technique in hydrodynamics [76,82].

Luo Jie et al. [83] took the flow around a two-dimensional cylinder as an example to verify the feasibility of the DMD analysis of unsteady flow fields. The base flow mode, the low-frequency convective dominant mode, and the high-frequency oscillation dominant mode under the limit cycle were extracted. It was revealed that the limit cycle state of the flow around a cylinder is formed by the superposition of the base flow state, the low-frequency convection state, and the high-frequency oscillation state. Ye Kun et al. [84] analyzed the Karman vortex street around a cylinder with the POD and DMD methods, found that the DMD method can accurately extract the main vortex structure and more high-order harmonic modes, and then obtained the characteristics of the substructure in time and space as orthogonal to each other. The main characteristics of the flow field in time and space can be obtained, and the stability of its extracted mode can also be analyzed. Williams MO et al. [85,86] studied the correction of the DMD method around the flow field when the Nyquist sampling theorem was not satisfied. The corrected DMD method can repair the missing flow field information within 5% to complete the analysis. For a special kind of flow field around a sphere, Williams MO also adopted the nuclear method to compare and analyze the information contained in the flow field and compared it with the flow field that was decomposed and reconstructed by the DMD method. Through the Koopman spectrum analysis method, it was found that the DMD method can be effective. Each mode expresses different characteristics of the flow field. Ponitz B et al. [87] used the PIV method to measure the evolution of the vortex ring within a certain period of time, used DMD to post-process the PIV time-resolved 3D data obtained in the experiment, and reconstructed the secondary structure of the vortex ring through the Q criterion. They found that the vortex ring and the core have azimuthal instability, and the vortex core grows gradually when the vortex ring n = 6.

DMD can quickly extract the main structure of the flow field, which is convenient for understanding the characteristics of the flow field and reconstructing the flow field. Sun Guoyong et al. [88] used DMD to reconstruct the two-dimensional flow field around the cylinder, and the error was less than 10^{-10}. However, for a more complex flow system, before using DMD to decompose the flow field, the complexity of the flow field should be analyzed to determine the amount of data to be obtained, to reduce errors, and to avoid distortion when reconstructing the flow field [89]. DMD can also be used for flow field predictions. The prediction error for a periodic flow field is only 10^{-4} [90], and DMD can predict the periodic flow field at any time. For the sampling interval of the unsteady flow field, the model constructed by the DMD method can predict the flow field within a certain period of time, but the error will increase with the enhancement of the nonlinear characteristics of the flow field [91].

4. The Future of Modal Decomposition

Mode decomposition has prospects for a wide application. On the one hand, modal decomposition has been applied to the aerodynamic simulation and control of aircraft and automobiles and in other fields, providing important support for the corresponding design and optimization. On the other hand, with the continuous development of computer hardware and algorithms, the computational cost of modal decomposition will become affordable, and it can be applied to more complex and large-scale flow simulations and controls.

4.1. Research Value and Advantages of Modal Decomposition

Modal decomposition should be explored in the following aspects. The first is to further improve the accuracy and reliability of modal decomposition and to reduce subjectivity and uncertainty. The second is to explore new modal decomposition methods, such as modal decomposition based on deep learning, to improve the efficiency and accuracy of modal decomposition. The third is to combine modal decomposition with other numerical simulation methods, such as multi-grid GPU programming and computing, etc., to achieve more efficient flow control and optimization in practical applications. As future engineering structural systems become more and more complex and nonlinear factors are coupled with each other, many simplified models are no longer used. Areas worthy of future attention include the following:

First, combined with the characteristics of each dimensionality reduction method, the second dimensionality reduction is performed on the high-dimensional system. The LS method can retain the topological properties of the original system. Therefore, the high-dimensional complex system can be reduced first by the POD method, and then by the LS method. Research into further dimensionality reductions not only maintains the convergence of the decomposition but also saves computing memory, which is very suitable for analyzing large data sets and is meaningful to any numerical research field.

Secondly, with the development of artificial intelligence, the intersection of machine learning and other disciplines, as well as efficient data processing methods, represents new trends. Machine learning requires a large amount of data for training, but its learning process relies on efficient methods of analysis for expressing features. The POD has advantages for processing data-driven machine learning.

Finally, most of the major breakthroughs and major innovations at the forefront of the disciplines are the result of interdisciplinary fusion and convergence. The intersection of the POD method and artificial intelligence fields, such as data-driven and deep learning, can make the POD method more widely used and more efficient in dimensionality reduction. For example, through using the POD method to reduce the dimension of nonlinear systems, using the POD method combined with wavelet transforms to identify damage and faults, using the POD method embedded in artificial intelligence algorithms to achieve reverse design, etc., the POD method has broad prospects in interdisciplinary fields.

4.2. Future Prospects for Mode Decomposition

Mode decomposition in fluid mechanics involves the decomposition of a complex flow field into some simple eigenmodes, in order to better understand and control the flow. Although modal decomposition has been widely used in the field of fluid mechanics, there are still some deficiencies.

First, the results of modal decomposition usually have a certain subjectivity and uncertainty. The parameters such as the basis function and the cut-off threshold selected for modal decomposition have a great influence on the results, so experience and thorough parameter study are needed to select appropriate parameters. In addition, due to the nonlinearity and complexity of the flow, the results of the modal decomposition may not be unique.

Second, modal decomposition cannot fully represent all the characteristics of the flow. Mode decomposition can only capture some important eigenmodes, but it cannot fully reflect all the information of the flow. Therefore, other factors and characteristics still need to be considered when simulating and controlling the flow.

Finally, modal decomposition is computationally expensive. In large-scale flow simulations and controls, modal decomposition requires a lot of computation and storage, resulting in a high computational cost. Therefore, in practical applications, it is necessary to balance the calculation cost of the modal decomposition and the requirement for results and to find a suitable balance point.

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